Supplement of

Monitoring of induced distributed double-couple sources using Marchenko-based virtual receivers

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S1 Classical homogeneous Green’s function representation

S1.1 Definition of the homogeneous Green’s function

Consider an inhomogeneous lossless acoustic medium with mass density \( \rho(x) \) and compressibility \( \kappa(x) \). In this medium, a space- and time-dependent source distribution \( q(x,t) \) is present, with \( q \) defined as the volume-injection rate density. The acoustic wave field, caused by this source distribution, is described in terms of the acoustic pressure \( p(x,t) \) and the particle velocity \( v_i(x,t) \). These field quantities obey the equation of motion and the stress-strain relation, according to

\[
\begin{align*}
\rho \partial_t v_i + \partial_i p &= 0, \\
\kappa \partial_t p + \partial_i v_i &= q.
\end{align*}
\] (S1, S2)

When \( q \) is an impulsive source at \( x = x_A \) and \( t = 0 \), according to

\[
q(x,t) = \delta(x - x_A) \delta(t),
\] (S3)

then the causal solution of Eqs. (S1) and (S2) defines the Green’s function, hence

\[
p(x,t) = G(x, x_A, t).
\] (S4)

By eliminating \( v_i \) from Eqs. (S1) and (S2) and substituting Eqs. (S3) and (S4), we find that the Green’s function \( G(x, x_A, t) \) obeys the following wave equation

\[
\partial_t (\rho^{-1} \partial_i G) - \kappa \partial^2_t G = -\delta(x - x_A) \partial_t \delta(t).
\] (S5)

Wave equation (S5) is symmetric in time, except for the source on the right-hand side, which is anti-symmetric. Hence, the time-reversed Green’s function \( G(x, x_A, -t) \) obeys the same wave equation, but with opposite sign for the source. By summing the wave equations for \( G(x, x_A, t) \) and \( G(x, x_A, -t) \), the sources on the right-hand sides cancel each other, hence, the homogeneous Green’s function

\[
G_h(x, x_A, t) = G(x, x_A, t) + G(x, x_A, -t)
\] (S6)

obeys the homogeneous equation

\[
\partial_t (\rho^{-1} \partial_i G_h) - \kappa \partial^2_t G_h = 0.
\] (S7)

S1.2 Reciprocity theorems

We define the temporal Fourier transform of a time-dependent quantity \( u(t) \) as

\[
u(\omega) = \int_{-\infty}^{\infty} u(t) \exp(i\omega t) dt.
\] (S8)

In the frequency domain, Eqs. (S1) and (S2) transform to

\[
\begin{align*}
-\iota \omega \rho v_i + \partial_i p &= 0, \\
-\iota \omega \kappa p + \partial_i v_i &= q.
\end{align*}
\] (S9, S10)

We introduce two independent acoustic states, which will be distinguished by subscripts A and B. Rayleigh’s reciprocity theorem is obtained by considering the quantity \( \partial_t \{ p_A v_i, B - v_i, A p_B \} \), applying the product rule for differentiation, substituting Eqs. (S9) and (S10) for both states, integrating the result over a spatial domain \( \mathbb{V} \) enclosed by surface \( \mathbb{S} \) with outward pointing
normal \( n_1 \), and applying the theorem of Gauss (de Hoop, 1988; Fokkema and van den Berg, 1993). Assuming that in \( V \) the medium parameters \( \rho(x) \) and \( \kappa(x) \) in the two states are identical, this yields Rayleigh’s reciprocity theorem of the convolution type

\[
\int_V \{ p_{AB} - q_{AP} \} \, dx = \int_S \frac{1}{i \omega \rho} \{ p_A (\partial_i p_B) - (\partial_i p_A) p_B \} n_i \, dx. \tag{S11}
\]

We derive a second form of Rayleigh’s reciprocity theorem for time-reversed wave fields. In the frequency domain, time-reversal is replaced by complex conjugation. When \( p \) is a solution of Eqs. (S9) and (S10) with source distribution \( q \) (and real-valued medium parameters), then \( p^* \) obeys the same equations with source distribution \(-q^*\). Making these substitutions for state A in Eq. (S11) we obtain Rayleigh’s reciprocity theorem of the correlation type (Bojarski, 1983)

\[
\int_V \{ p_{AB}^* + q_{AP}^* \} \, dx = \int_S \frac{1}{i \omega \rho} \{ p_A^* (\partial_i p_B) - (\partial_i p_A^*) p_B \} n_i \, dx. \tag{S12}
\]

### S1.3 Representation of the homogeneous Green’s function

We choose point sources in both states, according to \( q_A(x, \omega) = \delta(x - x_A) \) and \( q_B(x, \omega) = \delta(x - x_B) \), with \( x_A \) and \( x_B \) both in \( V \). The fields in states A and B are thus expressed in terms of Green’s functions, according to

\[
p_A(x, \omega) = G(x, x_A, \omega), \tag{S13}
\]
\[
p_B(x, \omega) = G(x, x_B, \omega), \tag{S14}
\]

with \( G(x, x_A, \omega) \) and \( G(x, x_B, \omega) \) being the Fourier transforms of \( G(x, x_A, t) \) and \( G(x, x_B, t) \), respectively. Making these substitutions in Eq. (S12) and using source-receiver reciprocity of the Green’s functions gives (Porter, 1970; Oristaglio, 1989; Wapenaar, 2004; Van Manen et al., 2005)

\[
G_h(x_B, x_A, \omega) = \int_S \frac{1}{i \omega \rho(x)} \left( \{ \partial_i G(x, x_B, \omega) \} G^*(x, x_A, \omega) - G(x, x_B, \omega) \partial_i G^*(x, x_A, \omega) \right) n_i \, dx, \tag{S15}
\]

where \( G_h(x_B, x_A, \omega) \) is the homogeneous Green’s function in the frequency domain. It is defined as

\[
G_h(x, x_A, \omega) = G(x, x_A, \omega) + G^*(x, x_A, \omega) = 2 \Re \{ G(x, x_A, \omega) \}, \tag{S16}
\]

where \( \Re \) denotes the real part. Equation (S15) is an exact representation for the homogeneous Green’s function \( G_h(x_B, x_A, \omega) \).

When \( S \) is sufficiently smooth and the medium outside \( S \) is homogeneous (with mass density \( \rho_0 \), compressibility \( \kappa_0 \) and propagation velocity \( c_0 = (\kappa_0 / \rho_0)^{-1/2} \), the two terms under the integral in Eq. (S15) are nearly identical (but opposite in sign), hence

\[
G_h(x_B, x_A, \omega) = -2 \int_S \frac{1}{i \omega \rho_0} G(x, x_B, \omega) \partial_i G^*(x, x_A, \omega) n_i \, dx. \tag{S17}
\]

The main approximation is that evanescent waves are neglected at \( S \) (Zheng et al., 2011; Wapenaar et al., 2011).

### S2 Single-sided homogeneous Green’s function representations

#### S2.1 Modification of the configuration

We replace the arbitrary closed surface \( S \) by a combination of two surfaces \( S_0 \) and \( S_A \), as indicated in Fig. S1. Here \( S_0 \) may be curved, but \( S_A \) is a horizontal surface, with \( n = (0, 0, 1) \). The depth level of \( S_A \) is defined as \( x_3, A \) (which is equal to
For state B we consider the Green’s function \( G \) which is identical to the actual medium in imaging across the scales (Wapenaar et al., 2019). This focusing function is defined in a truncated version of the medium, and, ignoring evanescent waves, plane.

\[
\begin{align*}
\int_{V_A} \{ p_A q_B - q_A p_B \} \, dx &= \int_{S_0} \frac{1}{i \omega \rho} \{ p_A (\partial_3 p_B) - (\partial_3 p_A) p_B \} n_i \, dx + \int_{S_A} \frac{1}{i \omega \rho} \{ p_A (\partial_3 p_B) - (\partial_3 p_A) p_B \} \, dx \\
&\quad \text{(S18)}
\end{align*}
\]

Following a similar derivation as in Appendix B in Wapenaar and Berkhout (1989), we reformulate Eqs. (S18) and (S19) as

\[
\begin{align*}
\int_{V_A} \{ p_A q_B - q_A p_B \} \, dx &= \int_{S_0} \frac{1}{i \omega \rho} \left( p_A (\partial_3 p_B) - (\partial_3 p_A) p_B \right) n_i \, dx - \int_{S_A} \frac{2}{i \omega \rho} \left( (\partial_3 p_A^+) p_B^- + (\partial_3 p_A^-) p_B^+ \right) \, dx \\
&\quad \text{(S20)}
\end{align*}
\]

The superscripts + and − stand for downgoing and upgoing, respectively. For state A we consider the focusing function \( f_1(x, x_A, \omega) = f_1^+(x, x_A, \omega) + f_1^-(x, x_A, \omega) \), introduced in section 3.1 in the companion paper “Green’s theorem in seismic imaging across the scales” (Wapenaar et al., 2019). This focusing function is defined in a truncated version of the medium, which is identical to the actual medium in \( V_A \), but reflection free above \( S_0 \) and below \( S_A \). The focusing conditions at the focal plane \( S_A \) are (Wapenaar et al., 2014)

\[
\begin{align*}
[\partial_3 f_1^+(x, x_A, \omega)]_{x_3=x_3,A} &= \frac{1}{2} i \omega \rho (x_A) \delta (x_H - x_{H,A}), \quad \text{(S22)}
\end{align*}
\]

\[
[\partial_3 f_1^-(x, x_A, \omega)]_{x_3=x_3,A} = 0. \quad \text{(S23)}
\]

For state B we consider the Green’s function \( G(x, x_B, \omega) = G^+(x, x_B, \omega) + G^-(x, x_B, \omega) \), with its source at \( x_B \) anywhere in the half-space below \( S_0 \). Note that the superscripts + and − in \( f_1^\pm(x, x_A, \omega) \) and \( G^\pm(x, x_B, \omega) \) refer to the propagation.
direction (downward or upward) at the observation point \(x\). The source of the Green’s function at \(x_B\) is omnidirectional. Substituting \(q_A(x, \omega) = 0\), \(p_A^+ (x, \omega) = f_1^+ (x, x_A, \omega)\), \(q_B(x, \omega) = \delta(x - x_B)\) and \(p_B^+ (x, \omega) = G^+ (x, x_B, \omega)\) into Eqs. (S20) and (S21), using Eqs. (S22) and (S23), gives

\[
G^-(x_A, x_B, \omega) + \chi(x_B) f_1(x_B, x_A, \omega) = \int_{S_0} \frac{1}{i \omega \rho(x)} \left( \{ \partial_t G(x, x_B, \omega) \} f_1(x, x_A, \omega) - G(x, x_B, \omega) \partial_t f_1(x, x_A, \omega) \right) n_i \, dx 
\]

(S24)

and

\[
G^+(x_A, x_B, \omega) - \chi(x_B) f_1^*(x_B, x_A, \omega) = - \int_{S_0} \frac{1}{i \omega \rho(x)} \left( \{ \partial_t G(x, x_B, \omega) \} f_1^*(x, x_A, \omega) - G(x, x_B, \omega) \partial_t f_1^*(x, x_A, \omega) \right) n_i \, dx, 
\]

(S25)

respectively, where \(\chi\) is the characteristic function of the domain \(V_A\). It is defined as

\[
\chi(x_B) = \begin{cases} 
1, & \text{for } x_B \text{ between } S_0 \text{ and } S_A, \\
\frac{1}{2}, & \text{for } x_B \text{ on } S = S_0 \cup S_A, \\
0, & \text{for } x_B \text{ outside } S.
\end{cases} 
\]

(S26)

Summing Eqs. (S24) and (S25) and using source-receiver reciprocity for the Green’s function on the left-hand side yields

\[
G(x_B, x_A, \omega) + \chi(x_B) 2i \Im \{ f_1(x_B, x_A, \omega) \} = \int_{S_0} \frac{2}{\omega \rho(x)} \left( \{ \partial_t G(x, x_B, \omega) \} \Im \{ f_1(x, x_A, \omega) \} - G(x, x_B, \omega) \Im \{ \partial_t f_1(x, x_A, \omega) \} \right) n_i \, dx, 
\]

(S27)

where \(\Im\) denotes the imaginary part. Taking the real part of both sides of this equation, using Eq. (S16), gives the single-sided representation of the homogeneous Green’s function

\[
G_h(x_B, x_A, \omega) = \int_{S_0} \frac{2}{\omega \rho(x)} \left( \{ \partial_t G_h(x, x_B, \omega) \} \Im \{ f_1(x, x_A, \omega) \} - G_h(x, x_B, \omega) \Im \{ \partial_t f_1(x, x_A, \omega) \} \right) n_i \, dx. 
\]

(S28)

S2.3 Single-sided homogeneous Green’s function representation: assuming a homogeneous upper half-space

From here onward we assume that also \(S_0\) is a horizontal surface, with \(n = (0, 0, -1)\). Following a similar derivation as in Appendix B in Wapenaar and Berkhout (1989), we reformulate Eqs. (S18) and (S19) as

\[
\int_{V_A} \left( p_A^+ q_B^- + p_A^- q_B^+ - q_A^+ p_B^- - q_A^- p_B^+ \right) \, dx = 
\]

\[
\int_{S_0} \frac{2}{i \omega \rho(x)} \left( (\partial_3 p_A^+) p_B^- + (\partial_3 p_A^-) p_B^+ \right) \, dx - \int_{S_A} \frac{2}{i \omega \rho(x)} \left( (\partial_3 p_A^+) p_B^- + (\partial_3 p_A^-) p_B^+ \right) \, dx 
\]

(S29)

and, ignoring evanescent waves,

\[
\int_{V_A} \left( p_A^+ q_B^- + p_A^- q_B^+ + q_A^+ p_B^- + q_A^- p_B^+ \right) \, dx = 
\]

\[
\int_{S_0} \frac{2}{i \omega \rho(x)} \left( (\partial_3 p_A^+) p_B^- + (\partial_3 p_A^-) p_B^+ \right) \, dx - \int_{S_A} \frac{2}{i \omega \rho(x)} \left( (\partial_3 p_A^+) p_B^- + (\partial_3 p_A^-) p_B^+ \right) \, dx. 
\]

(S30)
We apply these theorems to the situation in which the upper half-space above \( S_0 \) is homogeneous (for the Green’s function as well as for the focusing function). For state A we consider again the focusing function \( f_1(x, x_A, \omega) = f_1^+(x, x_A, \omega) + f_1^-(x, x_A, \omega) \), defined in a truncated version of the medium. For state B we consider the Green’s function \( G(x, x_B, \omega) = G^{+,-}(x, x_B, \omega) + G^{+,-}(x, x_B, \omega) + G^{+,-}(x, x_B, \omega) + G^{+,-}(x, x_B, \omega) \), with its source at \( x_B \) anywhere in the half-space below \( S_0 \). Note that we introduced two superscripts. The first superscript refers again to the propagation direction at the observation point \( x \). The second superscript refers to the radiation direction of the source at \( x_B \). Substituting \( q_A^+(x, \omega) = q_A^+(x, \omega) = 0 \), \( p_A^+(x, \omega) = f_1^+(x, x_A, \omega) \), \( q_B^+(x, \omega) = \delta(x - x_B) \), \( q_B^+(x, \omega) = 0 \) and \( p_B^+(x, \omega) = G^{+,-}(x, x_B, \omega) \) into Eqs. (S29) and (S30), using Eqs. (S22) and (S23) and \( G^{+,-}(x, x_B, \omega) = 0 \) for \( x \) at \( S_0 \) (since the upper half-space is homogeneous), gives

\[
G^{+,-}(x_A, x_B, \omega) + \chi(x_B)f_1^-(x_B, x_A, \omega) = \int_{S_0} \frac{2}{i\omega\rho_0} G^{+,-}(x, x_B, \omega) \partial_3 f_1^+(x, x_A, \omega) dx \tag{S31}
\]

and

\[
G^{+,-}(x_A, x_B, \omega) - \chi(x_B)f_1^+(x_B, x_A, \omega) = -\int_{S_0} \frac{2}{i\omega\rho_0} G^{+,-}(x, x_B, \omega) \partial_3 f_1^-(x, x_A, \omega) dx. \tag{S32}
\]

Next, substituting \( q_A^+(x, \omega) = q_A^+(x, \omega) = 0 \), \( p_A^+(x, \omega) = f_1^+(x, x_A, \omega) \), \( q_B^+(x, \omega) = 0 \), \( q_B^-(x, \omega) = \delta(x - x_B) \) and \( p_B^+(x, \omega) = G^{+,-}(x, x_B, \omega) \) into Eqs. (S29) and (S30), using Eqs. (S22) and (S23) and \( G^{+,-}(x, x_B, \omega) = 0 \) for \( x \) at \( S_0 \), gives

\[
G^{+,-}(x_A, x_B, \omega) + \chi(x_B)f_1^+(x_B, x_A, \omega) = \int_{S_0} \frac{2}{i\omega\rho_0} G^{+,-}(x, x_B, \omega) \partial_3 f_1^+(x, x_A, \omega) dx \tag{S33}
\]

and

\[
G^{+,-}(x_A, x_B, \omega) - \chi(x_B)f_1^-(x_B, x_A, \omega) = -\int_{S_0} \frac{2}{i\omega\rho_0} G^{+,-}(x, x_B, \omega) \partial_3 f_1^-(x, x_A, \omega) dx. \tag{S34}
\]

Summing Eqs. (S31) – (S34), using source-receiver reciprocity for the Green’s function on the left-hand side and \( G^{+,-}(x, x_B, \omega) = G^{+,-}(x, x_B, \omega) = 0 \) for \( x \) at \( S_0 \), we obtain

\[
G(x_B, x_A, \omega) + \chi(x_B)2i\Im \{f_1(x_B, x_A, \omega)\} = \int_{S_0} \frac{2}{i\omega\rho_0} G(x, x_B, \omega) \partial_3 (f_1^+(x, x_A, \omega) - \{f_1^-(x, x_A, \omega)\}) dx. \tag{S35}
\]

Taking the real part of both sides gives the single-sided representation of the homogeneous Green’s function for the situation that the upper half-space is homogeneous

\[
G_h(x_B, x_A, \omega) = 4\Re \int_{S_0} \frac{1}{i\omega\rho_0} G(x, x_B, \omega) \partial_3 (f_1^+(x, x_A, \omega) - \{f_1^-(x, x_A, \omega)\}) dx. \tag{S36}
\]

We conclude by deriving source-receiver reciprocity relations for the decomposed Green’s functions \( G^{\pm,\pm}(x, x_B, \omega) \). We consider Eq. (S29), but replace \( V_A \) by the entire space \( \mathbb{R}^3 \). In this situation there are only outgoing waves at \( S \). Hence, Eq. (S29) simplifies to

\[
\int_{\mathbb{R}^3} (p_A^+ q_B^- + p_A^- q_B^+ - q_A^+ p_B^- - q_A^- p_B^+) dx = 0. \tag{S37}
\]

First we substitute \( q_A^+ = \delta(x - x_A) \), \( q_A^- = 0 \), \( p_A^+ = G^{+,+}(x, x_A, \omega) \), \( q_B^+ = \delta(x - x_B) \), \( q_B^- = 0 \) and \( p_B^+ = G^{+,+}(x, x_B, \omega) \). This gives

\[
G^{+,-}(x_B, x_A, \omega) = G^{+,-}(x_A, x_B, \omega). \tag{S38}
\]
Next, we substitute \( q_A^+ = \delta(x - x_A), q_A^- = 0, p_A^\pm = G_{\pm,+}(x, x_A, \omega) \), \( q_B^+ = 0, q_B^- = \delta(x - x_B) \) and \( p_B^\pm = G_{\pm,-}(x, x_B, \omega) \). This gives

\[
G_{\pm,+}^+(x_B, x_A, \omega) = G_{\pm,-}^-(x_A, x_B, \omega). \tag{S39}
\]

Finally, we substitute \( q_A^+ = 0, q_A^- = \delta(x - x_A), p_A^\pm = G_{\pm,-}^+(x, x_A, \omega) \), \( q_B^+ = 0, q_B^- = \delta(x - x_B) \) and \( p_B^\pm = G_{\pm,-}^+(x, x_B, \omega) \). This gives

\[
G_{\pm,-}^+(x_B, x_A, \omega) = G_{\pm,-}^+(x_A, x_B, \omega). \tag{S40}
\]

Note that Eq. (S39) does not include a minus sign, unlike the corresponding relation for the flux-normalised decomposed Green’s functions (Wapenaar, 1996). As a result of this definition, we have the following simple expression for the full Green’s function

\[
G(x, x_A, \omega) = G_{\pm,+}^+(x, x_A, \omega) + G_{\pm,-}^+(x, x_A, \omega) + G_{\pm,+}^-(x, x_A, \omega) + G_{\pm,-}^-(x, x_A, \omega). \tag{S41}
\]
References


